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A two-step method to identify parameters of piecewise linear systems

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Abstract

Based on the direct parameter estimation method and the Legendre series approximation, a two-step method is presented to estimate by measured data the physical parameters, i.e., mass, damping, stiffness and knot, of nonlinear systems with symmetrical piecewise linear restoring force. At first, the piecewise linear restoring force of the system is approximated by Legendre series. A least squares direct parameter estimation method is adopted to identify the mass and damping parameters of the system and the corresponding coefficients of the Legendre series. Secondly, the physical parameters of the piecewise linear restoring force, i.e., the primary and the secondary stiffness and the knots, are extracted from the estimated Legendre coefficients. Neither iteration nor search process is needed in this method. The validity of the method is demonstrated by numerical simulation on a single-degree-of-freedom and a three-degree-of-freedom system with both noise-free data and noise-polluted data.

1. Introduction

Numerous methods have been developed for nonlinear system identification. However, very few works focused on piecewise linear (PWL) systems [1]. Kerschen et al. [2] gave a good review of some popular approaches for nonlinear system identification, some of which were developed to deal with general nonlinear models by approximating the nonlinearity with a finite sum of known basis functions with unknown coefficients, such as the restoring force surface (RFS) method [3], the direct parameter estimation (DPE) method [4] and the nonlinear autoregressive moving average with exogenous inputs (NARMAX) modeling method [5,6]. One disadvantage of these general methods is that they are not physical. In other words, the identified parameters usually have little or no physical meaning. Another disadvantage of these methods is that they tend to produce a very large number of model parameters, especially for systems with piecewise linear characters. The disadvantages limit the applicability of these methods to engineering problems.

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The RFS method for identifying nonlinear system parameters proposed by Masri and Caughey [3] assumes that the input/output data and the mass of the system were known. It describes the system restoring force over the displacement–velocity plane (i.e., phase plane) by an orthogonal polynomial series in the time domain. Masri et al. [7] extended the approach for parameter identification of multi-degree-of-freedom systems. In other extensions developed by Yang et al. [8] and Masri et al. [9], ordinary polynomial series are used instead of the orthogonal polynomial series. Mohammad et al. [4] presented the DPE method to directly estimate the physical parameters of general mechanical structures, which was discretized into lumped masses connected to each other and to the ground by linear or nonlinear restoring force links. Various types of restoring force links, including polynomial-formed stiffness and damping links, polynomial-formed cross-coupling nonlinearity links, Coulomb friction links, square-law damping links and piecewise linear links, were addressed in the model. Applications to systems with linear links, cubic nonlinearity links and/or Coulomb friction links were demonstrated by using singular value decomposition (SVD) method for unknown parameters estimation. It was shown that it was possible to identify all the physical parameters of a nonlinear multi-degree-of-freedom system by using only one input excitation, and that the RFS method could be the special case of the DPE as a byproduct. However, for cases without a priori information available about the location of the knots or turning points of PWL nonlinearity links, the DPE method may need to be modified by employing grid search method [1] or iteratively nonlinear least-squares technique as opposed to non-iteratively technique. The modification increases the complexity and difficulties to implement the algorithm for parameters estimation since the convergence of the iteration or grid searching should be ensured. Worden and Tomlinson [10] tried to identify an impacting cantilever beam with symmetrical PWL stiffness using the RFS method, but the results were not good because of poor instrumentation. Kerschen et al. [1] demonstrated numerical and experimental identifications of impacting cantilever beams with symmetrical or asymmetrical PWL stiffness using the RFS method. Two models, i.e., the polynomial model and the PWL model, were used to represent the PWL restoring force. For the polynomial model, least-squares method could be used directly to identify coefficients of the polynomial. For the PWL model on the other hand, the stiffness curve was plotted and inspected to find the approximate range of the knot and the grid search method was then utilized to increase the identification accuracy of the knot. Although the physical parameters can be identified in this way, the computing burden is very high since the inspection and grid search processes are involved. In fact, when the polynomial model is used to represent the nonlinear restoring forces, the identified polynomial coefficients are relevant to the parameters of the nonlinear restoring forces. It is thus possible to compute the parameters of the nonlinear restoring forces directly from the identified polynomial coefficients. To overcome the limitation of the above approach in determining the knots of the system, this paper uses Legendre polynomial series to approximate the symmetric piecewise restoring forces, and establishes the relationship between the polynomial coefficients identified by DPE method and the corresponding physical parameters of the restoring forces. For demonstration only systems with symmetric piecewise linear stiffness that often occur in engineering practice are considered. All the derivations and discussions will be focused on such systems.

2. Representation and normalization of piecewise linear functions

The mathematical model of a symmetrical PWL spring is given by

$$y(x) = \begin{cases} k_1 x, & |x| \leqslant x_s \\ k_2 x + (k_1 - k_2)x_s, & x_s < x \\ k_2 x - (k_1 - k_2)x_s, & x < -x_s \end{cases} \tag{1}$$

where x is the displacement, k_1 is the primary stiffness, k_2 is the secondary stiffness, x_s and $-x_s$ are the knots of the stiffness curve and $y(x)$ is the PWL restoring force. It is obvious that the PWL stiffness function is determined by three parameters, i.e. k_1, k_2 and x_s. The identification of the PWL nonlinearity is just to find these parameters. In the identification process, the measured displacement signal must be limited to a definite range of $[-x_d, x_d]$ (Fig. 1). Here x_d is a positive real number greater than x_s and is usually chosen as the absolute maximum value of the measured displacement. As a consequence, Eq. (1) is

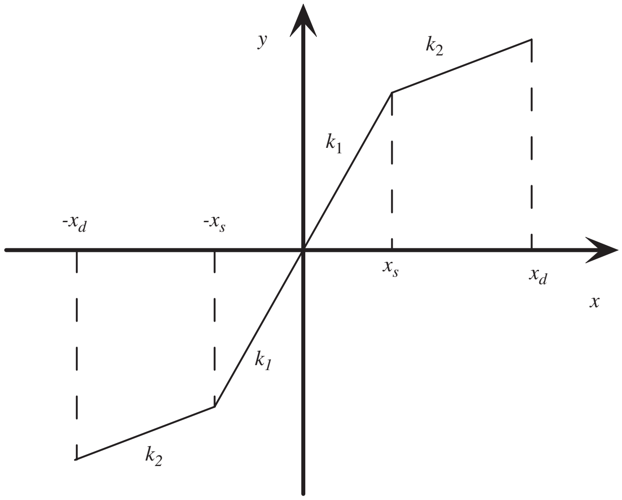

Fig. 1. Piecewise linear stiffness curve for identification use.

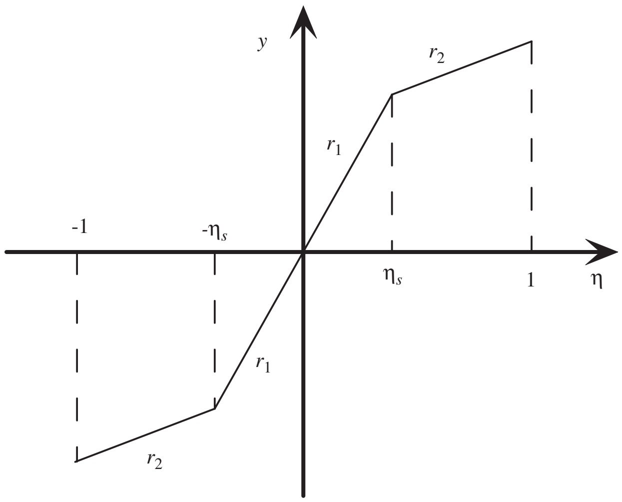

Fig. 2. Normalized piecewise linear stiffness curve.

rewritten as

$$y(x) = \begin{cases} k_1 x, & |x| \leqslant x_s \\ k_2 x + (k_1 - k_2)x_s, & x_s < x \leqslant x_d \\ k_2 x - (k_1 - k_2)x_s, & -x_d \leqslant x < -x_s \end{cases} \tag{2}$$

using the following transformation:

$$x = \eta x_d \tag{3}$$

renders Eq. (2) to be

$$y(\eta) = \begin{cases} r_1 \eta, & |\eta| \leqslant \eta_s \\ r_2 \eta + (r_1 - r_2)\eta_s, & \eta_s < \eta \leqslant 1 \\ r_2 \eta - (r_1 - r_2)\eta_s, & -1 \leqslant \eta < -\eta_s \end{cases} \tag{4}$$

where η is the normalized displacement, r_1 is the normalized primary stiffness, r_2 is the normalized secondary stiffness, r_s and $-r_s$ are the knots of the normalized stiffness curve, respectively, and $y(\eta)$ is the PWL restoring force (Fig. 2).

Once the normalized PWL parameters r_1, r_2 and η_s in Eq. (4) have been identified, the physical parameters can be obtained using the transformations as follows:

$$\begin{cases} k_1 = r_1/x_d \\ k_2 = r_2/x_d \\ x_s = \eta_s x_d \end{cases} \tag{5}$$

3. Fitting piecewise linear function with Legendre polynomials

In this section, Legendre polynomials will be used to fit the normalized PWL stiffness function in Eq. (4) by using the least-squares method. The relationships between the Legendre polynomial coefficients and the PWL parameters will be established as a sequel.

3.1. Legendre polynomials

The Legendre polynomials $P_n(x)$ can be expressed by the Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2-1)^n] \quad (n = 0, 1, 2, \ldots) \tag{6}$$

The first five Legendre's polynomials are:

$$1,\ x,\ (3x^2-1)/2,\ (5x^3-3x)/2,\ (35x^4-30x^2+3)/8. \tag{7}$$

Due to their orthogonality, Legendre polynomials $P_n(x),\ n = 0, 1, 2, 3\ldots$ form a complete orthogonal set on the interval $-1 \leqslant x \leqslant 1$, and satisfy

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \begin{cases} \dfrac{2}{2n+1}, & m = n \\ 0, & m \neq n \end{cases} \quad (m, n = 0, 1, 2, \ldots) \tag{8}$$

In addition, Legendre polynomials are symmetric or antisymmetric, i.e.,

$$P_n(-x) = (-1)^n P(x) \tag{9}$$

In other words, $P_n(x)$ is an even function when n is an even integer and an odd function when n takes an odd value.

3.2. Legendre series approximation to piecewise linear function

Eq. (4) is defined on the interval $[-1, 1]$, so Legendre polynomials can be used directly to approximate it in the least-squares sense. The relationship between the Legendre polynomial coefficients and the normalized parameters r_1, r_2 and η_s in Eq. (4) is easy to find as follows.

A function $y(\eta)$ defined on $[-1, 1]$ can be approximated by an n-order Legendre series as

$$S_n(\eta) = a_0P_0(\eta) + a_1P_1(\eta) + \cdots + a_nP_n(\eta) \tag{10}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are chosen to minimize the error

$$E = \frac{1}{2}\int_{-1}^{1} [S_n(\eta) - y(\eta)]^2\,\mathrm{d}\eta \tag{11}$$

which is achieved by letting

$$\frac{\partial E}{\partial a_i} = 0, \quad i = 0, 1, \ldots, n. \tag{12}$$

Substitute the equations resulted from inserting Eqs. (6) and (10) into Eq. (11), into Eq. (12) finally yields the Legendre coefficients for the function $y(\eta)$

$$a_i = \frac{\int_{-1}^{1} y(\eta) P_i(\eta)\,\mathrm{d}\eta}{\int_{-1}^{1} P_i(\eta) P_i(\eta)\,\mathrm{d}\eta} = \frac{2i+1}{2}\int_{-1}^{1} y(\eta) P_i(\eta)\,\mathrm{d}\eta \tag{13}$$

For a symmetrical PWL function, the Legendre coefficients can be obtained simply by substituting Eq. (4) into Eq. (13). Since Eq. (4) is an odd function, the even-order Legendre coefficients are equal to zeros, i.e., $a_{2k} = 0$ for $k = 0, 1, \ldots$. The odd-order Legendre coefficients are given by

$$a_{2k+1} = \begin{cases} \frac{1}{2}(r_1 - r_2)\eta_s(-\eta_s^2 + 3) + r_2, & k = 0 \\ (r_1 - r_2)\int_{\eta_s}^{1} (\eta_s - \eta) P_{2k+1}(\eta)\,\mathrm{d}\eta, & k = 1, 2, \ldots \end{cases} \tag{14}$$

To evaluate the fitness between the Legendre series and the PWL function, the fitness indicator (FI) is defined as

$$\mathrm{FI}(n) = 1 - \frac{\int_{-1}^{1} [S_n(\eta) - y(\eta)]^2\,\mathrm{d}\eta}{\int_{-1}^{1} y(\eta)^2\,\mathrm{d}\eta} \tag{15}$$

The closer the value of FI is to 1, the better is the fitness.

It can be seen from Eq. (14) that all the odd-order Legendre coefficients $a_{2k+1},\ k = 0, 1, 2, \ldots$ except a_1 can be expressed by the product of $r_1 - r_2$ and a $(2k+3)$-order polynomial of η_s. There are three unknown parameters, η_s, r_1 and r_2 to be identified in the PWL model, so at least three equations are needed. The equation corresponding to a_1 is definitely necessary, otherwise it is only possible to obtain $r_1 - r_2$ instead of r_1 and r_2 no matter how many equations are used. The order of η_s in a_{2k+1} is $2k+3$, so the larger k is, the higher is the order and the harder is it to solve the corresponding equations because high-order equations may have no analytical solution or have multi-solutions. Since the lower order polynomials usually contribute more to the function to be fitted, as can be seen from Fig. 3 of the single-degree-of freedom example in Section 5.1, the lower order coefficient equations will have better signal-to-noise-ratio (SNR) for extracting the PWL parameters. So the equations corresponding to a_1, a_3 and a_5 are chosen for the PWL parameters estimation in this paper.

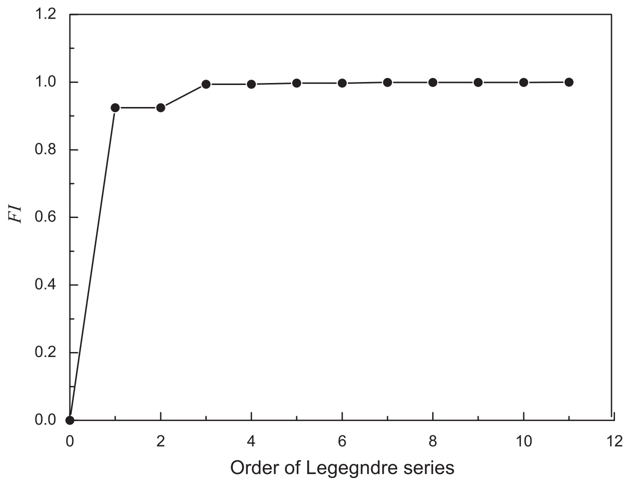

Fig. 3. Fitness of the nth order Legendre series for approximating the PWL function in Section 5.1.

From Eq. (14), a_1, a_3 and a_5 are deduced as follows:

$$a_1 = \frac{1}{2}(r_1 - r_2)\eta_s(-\eta_s^2 + 3) + r_2 \tag{16}$$

$$a_3 = -\frac{7}{8}(r_1 - r_2)\eta_s(1 - \eta_s^2)^2 \tag{17}$$

$$a_5 = \frac{11}{16}(r_1 - r_2)\eta_s(1 - 3\eta_s^2)(1 - \eta_s^2)^2 \tag{18}$$

The PWL parameters are then obtained

$$\eta_s = \sqrt{\frac{1}{3} + \frac{14a_5}{33a_3}} \tag{19}$$

$$r_1 = a_1 - \frac{4(\eta_s + 2)}{7\eta_s(1 + \eta_s)^2}a_3 \tag{20}$$

$$r_2 = a_1 + \frac{4(3 - \eta_s^2)}{7(1 - \eta_s^2)^2}a_3 \tag{21}$$

By means of these equations, the normalized knot value η_s and PWL stiffness r_1 and r_2 can be directly computed from the Legendre coefficients.

4. Identification procedure

Combing DPE method and equations in Sections 2 and 3 results in a two-step method for identifying the physical parameters of PWL systems as follows:

- *Step* 1. Approximate the PWL restoring forces by Legendre series with unknown coefficients and employ the DPE method to identify the parameters using the least-squares technique.
- *Step* 2. Substitute the identified Legendre coefficients into Eqs. (19)–(21) to get the normalized PWL parameters η_s, r_1 and r_2, then calculate the physical parameters x_s, k_1 and k_2 using Eq. (5).

4.1. Direct parameter estimation (DPE)

Direct parameter estimation method uses lumped parameter equations of motion as the identification model. For a general N degree-of-freedom system with an external force $u_i(t)$ applied at the ith degree-of-freedom, the equations of motion are

$$m_i\ddot{x}_i + \sum_{j=1}^{N} f_{ij}(\delta_{ij}, \dot{\delta}_{ij}) = u_i(t), \quad i = 1, \ldots, N \tag{22}$$

where m_i is the mass corresponding to the ith degree-of-freedom, x_i , $\dot{x}_i$ and $\ddot{x}_i$ are the displacement, velocity and acceleration of the ith degree-of-freedom, respectively, $\delta_{ij} = x_i - x_j$ ($\delta_{ii} = x_i$) and $\dot{\delta}_{ij} = \dot{x}_i - \dot{x}_j$ ($\dot{\delta}_{ii} = x_i$) are separately the relative displacement and velocity, and $f_{ij}(\delta_{ij}, \dot{\delta}_{ij})$ is the restoring force between the ith and the jth degrees of freedom.

A polynomial representation of f_{ij} renders Eq. (22) as

$$m_i\ddot{x}_i + \sum_{j=1}^{N}\sum_{k=0}^{p}\sum_{l=0}^{q} a_{(ij)kl}(\delta_{ij})^k(\dot{\delta}_{ij})^l = u_i(t) \tag{23}$$

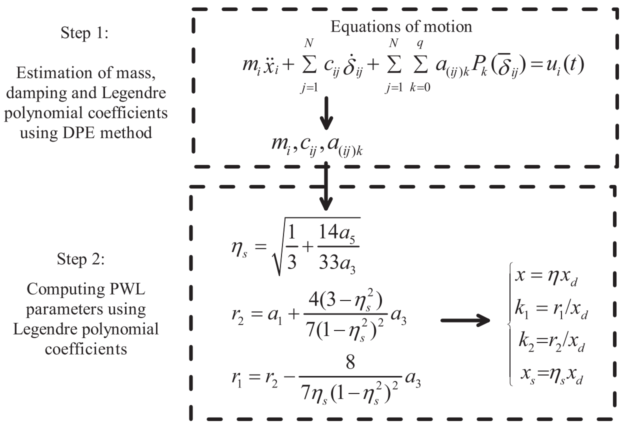

Fig. 4. The identification procedure.

with p and q as the highest orders of δ_{ij} and $\dot{\delta}_{ij}$ of the polynomial series, respectively. Once assuming that the system variables x_i, $\dot{x}_i$, $\ddot{x}_i$ and u_i are known at N_p sample data points, the equations of motion (23) can be solved for the unknown coefficients m_i and $a_{(ij)kl}$ in a least-squares sense, which best fit the measured data.

When the PWL restoring force of the system is approximated with a Legendre series, Eq. (23) becomes

$$m_i\ddot{x}_i+\sum_{j=1}^{N} c_{ij}\dot{\delta}_{ij}+\sum_{j=1}^{N}\sum_{k=0}^{p} a_{(ij)k}P_k(\overline{\delta}_{ij})=u_i(t) \tag{24}$$

where $\overline{\delta}_{ij}$ is the normalized relative displacement and $P_k(\overline{\delta}_{ij})$ is a k-order Legendre polynomial of $\overline{\delta}_{ij}$. As mentioned above, the least-squares technique can be used to identify the values of the coefficients m_i, c_{ij} and $a_{(ij)k}$.

4.2. Computation of the parameters of a piecewise linear function

Once estimated, $a_{(ij)1}$, $a_{(ij)3}$ and $a_{(ij)5}$ can be substituted into Eqs. (19)–(21) to extract the normalized PWL parameters $\eta_{(ij)s}$, $r_{(ij)1}$ and $r_{(ij)2}$. Eq. (5) is then used to calculate the physical parameters of the PWL restoring force between the ith and jth degree-of-freedom, i.e., $x_{(ij)s}$, $k_{(ij)1}$ and $k_{(ij)2}$.

The proposed two-step identification procedure is illustrated in Fig. 4.

5. Numerical simulation

5.1. A sdof system with piecewise linear stiffness

Consider the following sdof system with PWL stiffness [1]:

$$25\ddot{x}+15\dot{x}+f(x)=u(t) \tag{25}$$

where $u(t)$ is the excitation force, $f(x)$ is a PWL function given by Eq. (1) with the parameters $x_s=0.0004$, $k_1=330,000$ and $k_2=1,500,000$.

In [1] $u(t)$ is a band-limited white noise sequence, but in this paper harmonic excitation force $u(t)=600\sin(120\pi t)$ is adopted for the sake of simplicity. Details on the choice of excitation signal can be found in [11].

The system is numerically integrated in a time interval from 0 to 4 s using a fourth-order Runge–Kutta integration routine to generate the input and output data. The sampling frequency is set to 1000 Hz.

Only 1000 points of sampled data (data points from 2000 to 2999) are actually used in the identification procedure. It should be noted that the transient responses are included here to break the linear dependence between displacement and acceleration [11]. The time histories of the acceleration, velocity, displacement and excitation force are assumed being measured simultaneously for the identification. Both the noise-free case and the noise-polluted case are presented.

The mean-square error (MSE) [12] is adopted to reflect the error between the measured force value and the predicted force value . The definition is

$$\mathrm{MSE} = \frac{100}{N_p \sigma_u^2} \sum_{i=1}^{N_p} (u_i - \hat{u}_i)^2 \tag{26}$$

where N_p is the total number of samples used in the identification procedure, σ_u^2 is the variance of the measured force. The MSE can take on any value equal to or greater than zero, with a value closer to zero indicating a better fitness.

In the noise-free case, DPE method is used directly to identify the parameters of the system given by Eq. (25). As the PWL restoring force is an odd function, only odd-order Legendre polynomials are used in Eq. (24). Theoretical Legendre coefficients can be calculated by using Eq. (14) when the PWL restoring force is assumed known. Table 1 presents the theoretical and identified Legendre coefficients, and the corresponding MSE values, which vary with the highest order p of the Legendre series.

From the third row of Table 1, it is noted that the first three Legendre polynomials result in an MSE of 0.038%. However, such a small MSE value does not indicate a good estimation of the Legendre coefficients to the theoretical ones especially for a_3 and a_5. It is surprising to note that the identified Legendre coefficients change with p. Theoretically however, Eq. (13) indicates that the Legendre coefficients do not change along with p. Vectors composed of the measured discrete data instead of continuous functions are used in the DPE identification procedure and are not orthogonal to each other strictly, the identified coefficients will thus vary with p. In order to get a good estimation using only the first three Legendre coefficients, p should be carefully chosen.

Results of a large number of numerical simulations indicate that good agreement can be obtained when $p \geqslant 9$, while too big p increases computation complexity and even cause ill conditioning problem to implementation. So p is selected to 9 in this model.

Substitute a_1, a_3 and a_5 in each row of Table 1 into Eqs. (19)–(21) leads to the corresponding normalized PWL parameters η_s, r_1 and r_2, respectively. The physical parameters x_s, k_1 and k_2 of the PWL system can be then obtained by using Eq. (5). The actual and identified physical parameters with the corresponding MSE values are listed in Table 2. The identified PWL parameters are almost identical to the exact ones when $p = 9$ or 11.

The exact and identified stiffness curves are illustrated in Fig. 5. Both the polynomial- and the PWL-type stiffness curves are nearly identical to the exact one.

For the second case, measurements on the system excitation and responses are noise-polluted at a level of 5% of the corresponding root mean-square value of a white Gaussian noise signal. The noise-polluted signals are low-pass filtered before used for identification. The exact and identified Legendre coefficients with the corresponding MSE values are listed in Table 3 with different p. The MSE values in the noisy case are larger than the noise-free one, but similar conclusions can be reached by comparing Table 3 with Table 1.

Similar to the procedures adopted for noise-free case, the physical parameters x_s, k_1 and k_2 of the PWL system can be obtained. The actual and identified physical parameters with the corresponding MSE values are

Table 1
Identified coefficients for Legendre polynomials using DPE (noise free)

	a_1	a_3	a_5	a_7	a_9	a_{11}	MSE (%)
Theoretical coef.	303.17	127.07	32.83	−32.15	−11.30	17.66	
Identified coef. ($p = 5$)	306.62	136.91	49.99	–	–	–	0.038
Identified coef. ($p = 7$)	303.79	127.56	37.60	−29.06	–	–	0.018
Identified coef. ($p = 9$)	303.28	126.72	32.01	−36.50	−19.41	–	0.011
Identified coef. ($p = 11$)	302.96	126.79	31.61	−31.93	−13.92	17.51	0.006

Table 2
Identified piecewise linear parameters (noise free)

	x_s (m)	k_1 (N m^{-1})	k_2 (N m^{-1})	MSE (%)
Actual value	0.0004	330,000	1,500,000	
Identified value ($p = 5$)	0.000420	335,933	1,758,519	0.018
Identified value ($p = 7$)	0.000407	334,957	1,556,336	0.002
Identified value ($p = 9$)	0.000399	329,919	1,489,752	0.000
Identified value ($p = 11$)	0.000398	328,866	1,485,400	0.000

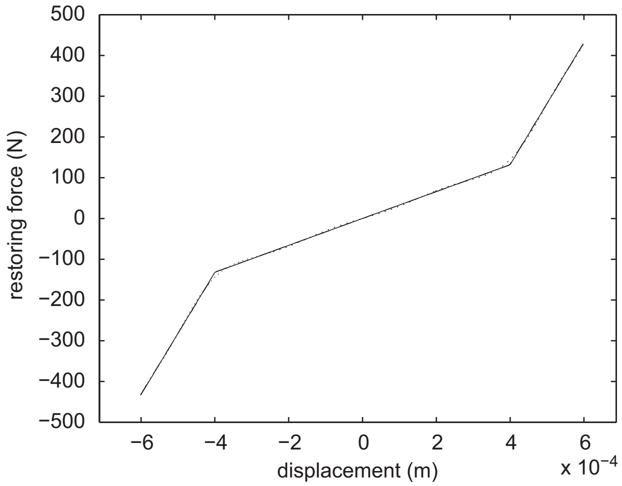

Fig. 5. Comparison between the real and reconstructed stiffness curves (noise free): —, exact; - - -, reconstructed (polyn.); · · ·, reconstructed (PWL).

Table 3
Identified coefficients for Legendre polynomials using DPE (5% noise)

	a_1	a_3	a_5	a_7	a_9	a_{11}	MSE (%)
Theoretical coef.	303.26	127.12	32.81	−32.17	−11.28	17.67	
Identified coef. ($p = 5$)	311.11	143.93	56.54	–	–	–	0.576
Identified coef. ($p = 7$)	308.45	135.44	45.07	−24.93	–	–	0.563
Identified coef. ($p = 9$)	307.46	133.40	38.33	−33.30	−18.44	–	0.557
Identified coef. ($p = 11$)	307.90	134.87	39.75	−27.38	−11.98	16.36	0.553

Table 4
Identified piecewise linear parameters (5% noise)

	x_s (m)	k_1 (N m^{-1})	k_2 (N m^{-1})	MSE (%)
Theoretical coef.	0.0004	330,000	1,500,000	
Identified coef. ($p = 5$)	0.000425	337,818	1,885,857	0.573
Identified coef. ($p = 7$)	0.000414	336,935	1,690,637	0.558
Identified coef. ($p = 9$)	0.000406	332,242	1,598,919	0.555
Identified coef. ($p = 11$)	0.000407	331,992	1,623,056	0.555

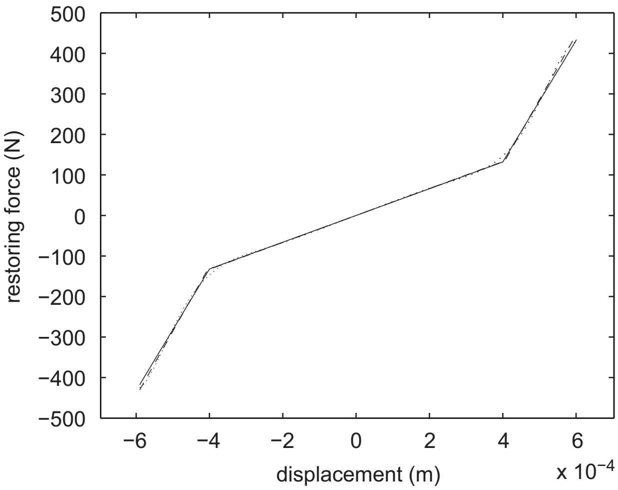

Fig. 6. Comparison between the real and reconstructed stiffness curves (5% noise): —, exact; - - -, reconstructed (polyn.); · · ·, reconstructed (PWL).

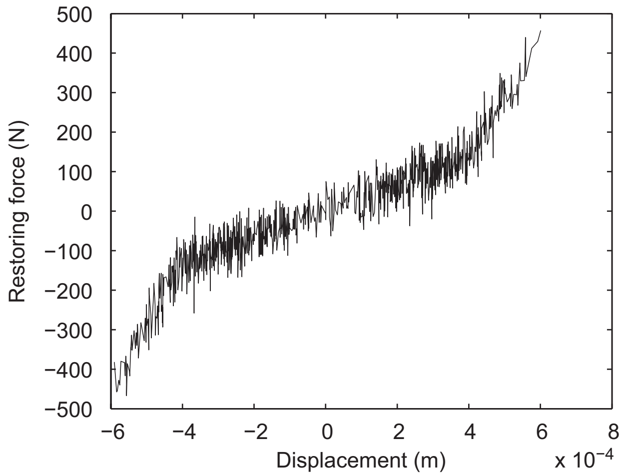

Fig. 7. Stiffness curve.

listed in Table 4. The identified PWL parameters are slightly worse than those obtained for the noise-free case, but they are still almost identical to the exact ones when $p = 9$ or 11.

Fig. 6 shows the exact and identified stiffness curves in the noisy case. Both the polynomial- and the PWL-type stiffness curves agree well with the exact one.

As a comparison, the method introduced by Kerschen et al. [1] is also used to identify the system. In this method, the stiffness curve, i.e., the measured restoring force versus the displacement, is needed and plotted in Fig. 7. Inspection of the stiffness curve indicates that the knot value occurs around 0.0004 m. Then grid search method is utilized to increase the accuracy. The MSE is computed for 200 knot values regularly spaced between 0.0003 and 0.0005 m. Fig. 8 presents the evolution of the MSE with the knot value. Optimum knot value turns out to be 0.000403 m with an MSE of 0.548%. The identified primary and secondary stiffnesses are 333,395 and 1,525,487 N/m, respectively. It is a somewhat better estimation than that we get using two-step method since its MSE is slightly smaller. However, the computing burden is much higher because inspection of

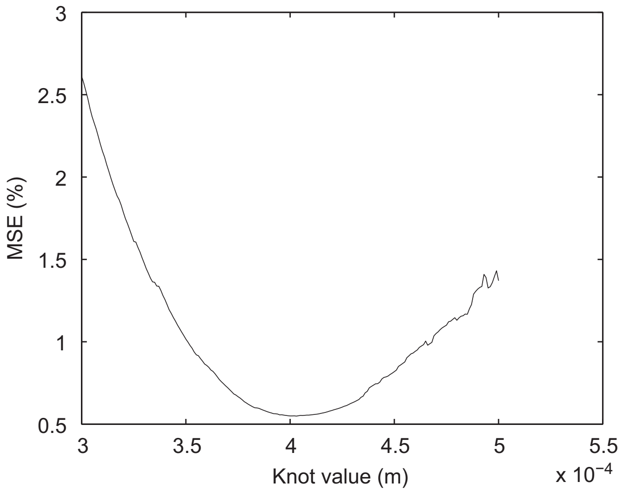

Fig. 8. MSE versus knot value.

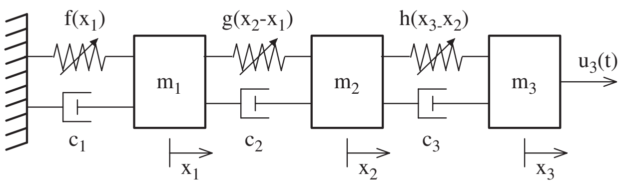

Fig. 9. 3dof spring mass system.

Table 5
Exact and identified piecewise linear parameters (noise free)

	x_{fs}	k_{f1}	k_{f2}	x_{gs}	k_{g1}	k_{g2}	x_{hs}	k_{h1}	k_{h2}
Exact	0.01	5000	3000	0.02	3000	2000	0.02	2000	4000
Identified (noise free)	0.0096	5115	3196	0.0187	3091	2071	0.0196	1986	4051
Identified (5% noise)	0.0095	5022	3243	0.0194	3113	1989	0.0188	1983	4041

the stiffness curve and grid search processes are involved. Actually, both methods provide a reliable estimation since the mean-square errors are around 0.55%.

5.2. A three-degree-of-freedom system with piecewise linear stiffness

In this section the proposed identification procedure is applied to a three-degree-of-freedom PWL system shown in Fig.9. The motion equations of the system are given by

$$\begin{cases} 50\ddot{x}_1 + 20\dot{x}_1 - 20(\dot{x}_2 - \dot{x}_1) + f(x_1) - g(x_2 - x_1) = 0 \\ 40\ddot{x}_2 + 20(\dot{x}_2 - \dot{x}_1) - 20(\dot{x}_3 - \dot{x}_2) + g(x_2 - x_1) - h(x_3 - x_2) = 0 \\ 20\ddot{x}_3 + 20(\dot{x}_3 - \dot{x}_2) + h(x_3 - x_2) = u_3 \end{cases} \tag{27}$$

where $u_3(t) = 40\sin(5\pi t)$ is a harmonic excitation force, $f(x_1)$, $g(x_2 - x_1)$ and $h(x_3 - x_2)$ are the PWL restoring forces of the springs. According to Mohammad et al. [4], the parameters can be identified with only one excitation location on the system. The parameters of the springs are listed in Table 5.

Table 6
Exact and identified system mass and damping parameters (noise free)

	m_1	c_1	m_2	c_2	m_3	c_3
Exact	50	10	40	10	20	10
Identified (noise free)	50.95	8.33	40.53	10.91	20.27	9.44
Identified (5% noise)	50.78	10.39	40.83	10.82	20.4	8.89

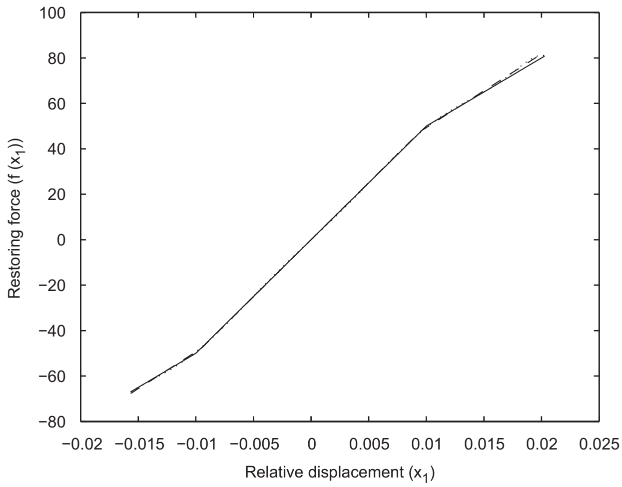

Fig. 10. Comparison between the real and reconstructed stiffness curves of the 3dof system (the first dof, 5% noise): —, exact; - - -, reconstructed (polyn.); · · ·, reconstructed (PWL).

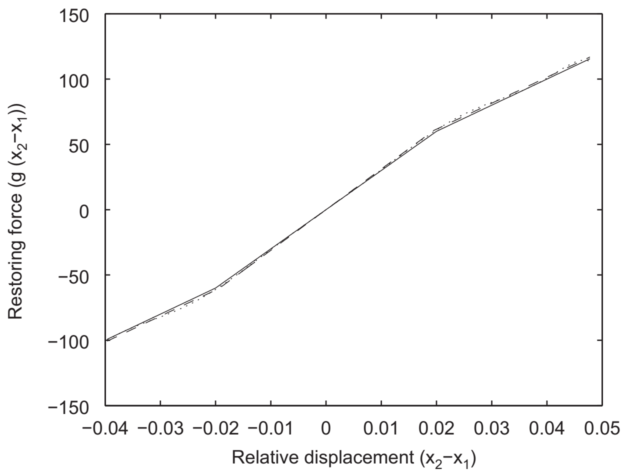

Fig. 11. Comparison between the real and reconstructed stiffness curves of the 3dof system (the second dof, 5% noise): —, exact; - - -, reconstructed (polyn.); · · ·, reconstructed (PWL).

The system is numerically integrated in a time interval from 0 to 10 s using a fourth-order Runge–Kutta integration routine to generate the input and output data. The sampling frequency is set to 500 Hz.

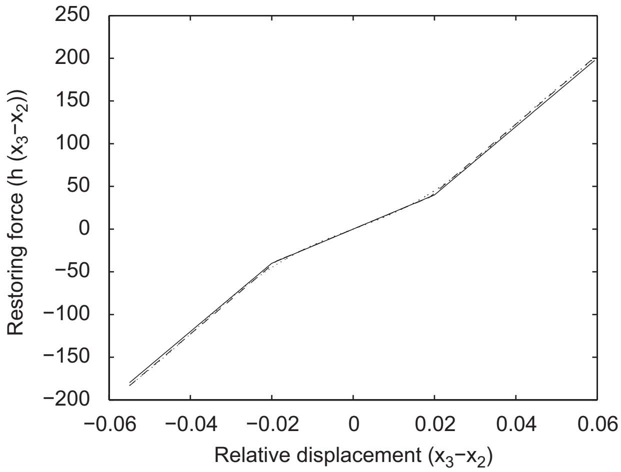

Fig. 12. Comparison between the real and reconstructed stiffness curves of the 3dof system (the third dof, 5% noise): —, exact; - - -, reconstructed (polyn.); $\cdots$, reconstructed (PWL).

Only 1000 points of the sampled data (data points from 501 to 1500) are actually used in the identification procedure. Both a noise-free case and a case of 5% noise level are considered. The identified mass, damping and PWL parameters are listed in Tables 6 and 5. $p = 9$ is adopted in the identification procedure.

From Tables 6 and 5 it can be seen that the proposed identification procedure can be easily extended to multi-degree-of-freedom PWL system. However, the accuracy of the identified parameters is worse than that in the sdof case. The MSE is 0.28% and 1.78% for the noise-free case and the noisy case, respectively.

Figs. 10–12 shows the exact and estimated stiffness curves of all the PWL springs in the noisy case. It should be noted that although the three-degree-of-freedom system has both hardening- and softening-type springs, the presented identification procedure still gives acceptable results.

6. Conclusions

A two-step procedure based on DPE method and Legendre series approximation has been presented for identifying physical parameters, i.e., mass, damping, stiffness and knot, of a single- or a multi-degree-of-freedom nonlinear system with symmetrical PWL restoring forces. The method takes the full advantage of the simplicity of the DPE method and the priori knowledge of the PWL system. Neither iteration nor search process is necessary in the method, and the physical parameters of the system can be obtained easily.

Numerical simulations for a single- and a three-degree-of-freedom systems using data without noise or with 5% noise level demonstrate that the proposed method performs well in the presence of noise and can deal with both hardening- and softening-type PWL nonlinearity. When the highest order of the Legendre polynomials is equal to 9, good agreement between the identified Legendre coefficients and the corresponding theoretical values can be obtained, and good estimation to the system physical parameters can be acquired as a sequence.

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